SOME RECURSIVE FORMULAS FOR SELBERG-TYPE INTEGRALS

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ABSTRACT. A set of recursive relations satisfied by Selberg-type integrals involving monomial symmetric polynomials are derived, generalizing previous results in [1, 11]. These formulas provide a well-defined algorithm for computing Selberg-Schur integrals whenever the Kostka numbers relating Schur functions and the corresponding monomial polynomials are explicitly known. We illustrate the usefulness of our results discussing some interesting examples.

1. INTRODUCTION

The Selberg integral and its generalizations have played a central role both in pure and applied mathematics. Their applications run from the proof of the Mehta-Dyson conjecture and several cases of the Macdonald conjectures [2, 20, 22] to the study of some q-analogues of constant term identities, through Calogero-Sutherland quantum many body models [7, 14, 15, 17, 27, 31], orthogonal polynomials theory [24, 28], hyperplane arrangements [25] and random matrix theory [4, 9, 18, 19]. They also have a deep connection to the Knizhnik-Zamolodchikov equations [23, 32] with the corresponding implications in conformal field theory and even string theory [5, 6, 8, 12, 13, 21, 29, 30]. See [10] for a comprehensive review on the relevance of the Selberg integral and its applications.

The aim of this paper is to study the Selberg-type integral with the integrand dressed up with a symmetric function, namely, to study integrals of the form $J_f \equiv J^{(N)}(a, b, \rho; f)$:

$$(1) \qquad J_f = \int_\Lambda f(y_1, \ldots, y_N) \prod_{i=1}^N y_i^{a-1}(1-y_i)^{b-1} \prod_{1 \le i < j \le N} |y_i - y_j|^{2\rho} dy_1 \wedge \cdots \wedge dy_N,$$

where $f(y_1, \ldots, y_N)$ is a symmetric polynomial, the integral is taken over the N-dimensional open domain[1] $\Lambda = (0,1)^N$ and a, b and ρ are complex numbers. For simplicity, we will denote the function $\prod_{i=1}^N y_i^{a-1}(1-y_i)^{b-1} \prod_{1 \le i < j \le N} |y_i - y_j|^{2\rho}$ by $\Phi(y)$, the N-form $dy_1 \wedge \cdots \wedge dy_N$ by dy and the polynomial $f(y_1, \ldots, y_N)$ by $f(y)$.

Among the basis for the space of symmetric polynomials, Schur basis plays a special role in this context. The importance of Selberg-Schur integrals was stated in [30] when studying the non-triviality of the integral representation of the intertwining operators between the Fock space representations of the Virasoro algebra and in [3], in a more general setting, when analyzing the Fock space resolutions of the $\widehat{sl}(n)$ irreducible highest-weight modules. As

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[1]Selberg-type integrals are sometimes defined over the N-dimensional simplex $\{(y_1, \ldots, y_N) \in \mathbb{R}^N \mid 0 < y_1 < \cdots < y_N < 1\}$. From the symmetry of the integrand under a permutation of the variables we get that these integrals differ by a factor of $1/N!$.

expected, they also appear when computing correlation functions on the sphere in related Wess-Zumino-Novikov-Witten models [12, 13]. Given a partition λ we will denote the Schur polynomial associated with it by $s_\lambda(y)$ and the corresponding Selberg-Schur integral by J_λ.

The case $\lambda = 0$ corresponds to the classical integral considered by Selberg in [26]. The analytic expression he found for this integral is:

$$(2) \qquad J_0 = \int_\Lambda \Phi(y) dy = \prod_{i=1}^{N} \frac{\Gamma(a + (N-i)\rho)\Gamma(b + (N-i)\rho)\Gamma(i\rho + 1)}{\Gamma(a + b + (2N - i - 1)\rho)\Gamma(\rho + 1)},$$

and it is well defined whenever a, b and ρ satisfy

$$(3) \qquad \Re(a), \Re(b) > 0 \qquad \text{and} \qquad \Re(\rho) > -\min\left\{\frac{1}{N}, \frac{\Re(a)}{N-1}, \frac{\Re(b)}{N-1}\right\},$$

the second inequality having meaning for $N > 1$. From now on we assume that these conditions always hold.

When $\lambda = (1^{m_1})$ with $0 \leq m_1 \leq N$, Schur polynomials reduce to elementary symmetric polynomials, *i.e.*,

$$(4) \qquad s_{(1^{m_1})}(y) \equiv e_{m_1}(y) = \frac{1}{N!}\binom{N}{m_1}\sum_{\sigma \in S_N}\prod_{i=1}^{m_1} y_{\sigma(i)},$$

where S_N is the set of permutations of the set $\{1, 2, \ldots, N\}$ and $e_0(y) = 1$. In this case, Aomoto [1] showed that

$$(5) \qquad J_{(1^{m_1})} = \int_\Lambda e_{m_1}(y)\Phi(y) dy = J_0\binom{N}{m_1}\prod_{i=1}^{m_1}\frac{a + (N-i)\rho}{a + b + (2N - 1 - i)\rho}.$$

A further extension of Selberg integral, by far the most general one, has been computed by Kadell in [16] and it involves Jack functions. It reads:

$$(6) \qquad \int_\Lambda P_\lambda^{(1/\rho)}(y)\Phi(y) dy = J_0 P_\lambda^{(1/\rho)}(1^N)\frac{[a + (N-1)\rho]_\lambda^{(\rho)}}{[a + b + 2(N-1)\rho]_\lambda^{(\rho)}},$$

where λ is an arbitrary partition, $P_\lambda^{(1/\rho)}(y)$ is a Jack polynomial and $[a]_\lambda^{(\rho)}$ is a generalized Pochhammer symbol, which is defined as

$$(7) \qquad [a]_\lambda^{(\rho)} = \prod_{i \geq 1}(a + (1-i)\rho)_{\lambda_i},$$

$(a)_n$ being the standard Pochhammer symbol, namely, $(a)_n = a(a+1)\cdots(a+n-1)$ with $(a)_0 = 1$. When $\rho = 1$ we have $P_\lambda^{(1/\rho)}(y) = s_\lambda(y)$ so that (6) gives

$$(8) \qquad J_\lambda = J_0 s_\lambda(1^N)\frac{[a + (N-1)]_\lambda^{(1)}}{[a + b + 2N - 2]_\lambda^{(1)}}.$$

More recently, it was proved in [11] that for the case $\lambda = (2^{m_2} 1^{m_1})$, $0 \leq m_1 + m_2 \leq N$, one has

$$
(9) \quad J_{(2^{m_2} 1^{m_1})} = J_0 m_\lambda(1^N) \frac{[a + (N-1)\rho]_\lambda^{(\rho)}}{[a + b + 2(N-1)\rho]_\lambda^{(\rho)}} \frac{[a + b + (N-2)\rho]_{(1^{m_2})}^{(\rho)}}{[a + b + (2N - m_1 - m_2 - 2)\rho]_{(1^{m_2})}^{(\rho)}}
$$
$$
\times {}_4F_3 \left[\begin{array}{c} -m_2, -N + m_1 + m_2, \alpha + \beta + \gamma + 2N - m_2 - 1, \alpha + N - m_2 + 1 \\ \alpha + \beta + N - m_2 - 1, \alpha + \gamma + N - m_2, m_1 + 2 \end{array} \right],
$$

where $\alpha = a/\rho$, $\beta = b/\rho$, $\gamma = 1/\rho$, the hypergeometric series ${}_4F_3$ is evaluated at 1 and $m_\lambda(y)$ denotes the monomial symmetric polynomial associated to the partition λ.

In this paper, using similar techniques as those employed in [1, 11], we find a set of recursive formulas satisfied by generic Selberg-type integrals involving monomial polynomials. These recursions and the fact that Schur polynomials can be uniquely decomposed as linear combinations of monomial symmetric functions reduce the problem of computing (1) to the problem of computing Kostka numbers while providing a well-defined algorithm for obtaining Selberg-Schur integrals in the general case.

The paper is organized as follows. After introducing some notation we prove in Section 2 some lemmas and preliminary propositions that will be useful for obtaining in Section 3 the recursive relations we have already announced. In Section 4 we illustrate the usefulness of our results with several relevant examples.

2. NOTATION AND PRELIMINARY LEMMAS

In this section we fix our conventions, we introduce some notation and we derive several formulas that will be needed in order to prove our main results.

Despite of the fact that partitions are usually defined without trivial components, it will be useful for our purposes to identify partitions with length $\ell_\lambda \leq N$ with decreasingly ordered N-tuples with non-negative entries by defining $\lambda_i = 0$ for $i = \ell_\lambda + 1, \ldots, N$.

Given $v \in \mathbb{R}^N$, $v = (v_1, \ldots, v_N)$, we define its (decreasingly) ordered partner $[v]$ as the vector $(v_{\sigma(1)}, \ldots, v_{\sigma(N)})$, where $\sigma \in S_N$ is any permutation satisfying $v_{\sigma(1)} \geq v_{\sigma(2)} \geq \cdots \geq v_{\sigma(N)}$. If $v_1, \ldots, v_N$ are all non-negative integer numbers, then $[v]$ actually defines a partition with length $\ell_{[v]} \leq N$. We denote the standard basis of $\mathbb{R}^N$ by $\{e_1, \ldots, e_N\}$, e_j being the j-th unit vector.

For a partition λ let us denote by λ' its conjugate so that λ'_k gives the number of entries $\geq k$ in λ. Notice that $\ell_\lambda = \lambda'_1$. If $y = (y_1, \ldots, y_N)$ we define

$$
(10) \qquad y^\lambda = \prod_{j=1}^N y_j^{\lambda_j} = \prod_{i=0}^n \prod_{r=1}^{m_{n-i}} y_{\lambda'_{n-i+1}+r}^{n-i},
$$

where n is the greatest part of λ, $m_k = \lambda'_k - \lambda'_{k+1}$, $k = 1, \ldots, n$, is the multiplicity of the part k in λ and $m_0 = N$. Further, let us define the following integrals:

$$
(11) \qquad B_\lambda = \int_\Lambda y^\lambda \Phi(y) dy,
$$

and, for any integer number $c \geq 0$,

$$(12) \qquad A_\lambda(k,c) = \int_\Lambda \frac{y_1^c \prod_{j=2}^N y_j^{\lambda_j}}{y_1 - y_k} \Phi(y) dy,$$

and

$$(13) \qquad K_\lambda(c) = \int_\Lambda \frac{y_1^c \prod_{j=2}^N y_j^{\lambda_j}}{1 - y_1} \Phi(y) dy.$$

We will denote $A_\lambda(k, \lambda_1)$ simply by $A_\lambda(k)$.

We will generalize [11, Lemma 1] and [11, Lemma 3] by proving the following.

Lemma 1. *Let λ be a partition such that $\ell_\lambda \leq N$ and let c be a non-negative integer number. Let $2 \leq k \leq N$. Then,*

$$(14) \qquad A_\lambda(k,c) = \begin{cases} -\frac{1}{2} \sum_{i=0}^{\lambda_k - 1 - c} B_{[\lambda + (c+i-\lambda_1)e_1 - (1+i)e_k]} & if \quad c < \lambda_k, \\ 0 & if \quad c = \lambda_k, \\ \frac{1}{2} \sum_{i=0}^{c - \lambda_k - 1} B_{[\lambda + (c-1-i-\lambda_1)e_1 + ie_k]} & if \quad c > \lambda_k. \end{cases}$$

Proof. Exchanging y_k and y_1 in (12) and then using the symmetry of Selberg's kernel $\Phi(y)$ under the permutation of any pair of variables, we obtain

$$(15) \qquad A_\lambda(k,c) = -\int_\Lambda \frac{y_1^{\lambda_k} y_k^c \prod_{j \neq 1,k}^N y_j^{\lambda_j}}{y_1 - y_k} \Phi(y) dy.$$

Thus, if $0 \leq c < \lambda_k$, we get

$$(16) \qquad A_\lambda(k,c) = -\frac{1}{2} \int_\Lambda \frac{y_1^c y_k^c (y_1^{\lambda_k - c} - y_k^{\lambda_k - c}) \prod_{j \neq 1,k}^N y_j^{\lambda_j}}{y_1 - y_k} \Phi(y) dy,$$

which is equivalent to

$$(17) \qquad A_\lambda(k,c) = -\frac{1}{2} \sum_{i=0}^{\lambda_k - c - 1} B_{[\lambda + (c+i-\lambda_1)e_1 - (1+i)e_k]},$$

where we have used

$$(18) \qquad y_1^{\lambda_k - c} - y_k^{\lambda_k - c} = (y_1 - y_k) \sum_{i=0}^{\lambda_k - c - 1} y_1^{\lambda_k - c - 1 - i} y_k^i.$$

When $c = \lambda_k$ it is straightforward to see that integral (12) vanishes.

If, instead, $c > \lambda_k$, then

$$(19) \qquad A_\lambda(k,c) = \frac{1}{2} \int_\Lambda \frac{y_1^{\lambda_k} y_k^{\lambda_k} (y_1^{c - \lambda_k} - y_k^{c - \lambda_k}) \prod_{j \neq 1,k}^N y_j^{\lambda_j}}{y_1 - y_k} \Phi(y) dy,$$

namely,

$$(20) \qquad A_\lambda(k,c) = \frac{1}{2} \sum_{i=0}^{c - \lambda_k - 1} B_{[\lambda + (c-1-i-\lambda_1)e_1 + ie_k]},$$

as we wanted to prove. □

Corollary 2. *Let λ be a partition with $\ell_\lambda \leq N$ and let $2 \leq k \leq N$. Then,*

$$(21) \qquad A_\lambda(k) = \frac{1}{2} \sum_{i=0}^{\lambda_1 - \lambda_k - 1} B_{[\lambda - (1+i)e_1 + ie_k]}.$$

Example 3. *If $\lambda = (2^{m_2} 1^{m_1})$, $m_1 + m_2 \leq N$, then*

$$(22) \qquad A_{(2^{m_2} 1^{m_1})}(k) = \begin{cases} 0 & \text{if } 2 \leq k \leq \lambda_2', \\ \frac{1}{2} B_{(2^{m_2-1} 1^{m_1+1})} & \text{if } \lambda_2' + 1 \leq k \leq \lambda_1', \\ B_{(2^{m_2-1} 1^{m_1+1})} & \text{if } \lambda_1' + 1 \leq k \leq N, \end{cases}$$

while Lemma 1 gives, for $c = 1$,

$$(23) \qquad A_{(2^{m_2} 1^{m_1})}(k, 1) = \begin{cases} -\frac{1}{2} B_{(2^{m_2-2} 1^{m_1+2})} & \text{if } 2 \leq k \leq \lambda_2', \\ 0 & \text{if } \lambda_2' + 1 \leq k \leq \lambda_1', \\ \frac{1}{2} B_{(2^{m_2-1} 1^{m_1-1})} & \text{if } \lambda_1' + 1 \leq k \leq N. \end{cases}$$

as it was already shown in [11, Lemma 1] and [11, Lemma 3], respectively. Furthermore, we find

$$(24) \qquad A_{(2^{m_2} 1^{m_1})}(k, 0) = \begin{cases} -B_{(2^{m_2-2} 1^{m_1+1})} & \text{if } 2 \leq k \leq \lambda_2', \\ -\frac{1}{2} B_{(2^{m_2-1} 1^{m_1-1})} & \text{if } \lambda_2' + 1 \leq k \leq \lambda_1', \\ 0 & \text{if } \lambda_1' + 1 \leq k \leq N. \end{cases}$$

Example 4. *If $\lambda = (3^{m_3} 2^{m_2} 1^{m_1})$, $\ell_\lambda \leq N$, then*

$$(25) \qquad A_{(3^{m_3} 2^{m_2} 1^{m_1})}(k) = \begin{cases} 0 & \text{if } 2 \leq k \leq \lambda_3', \\ \frac{1}{2} B_{(3^{m_3-1} 2^{m_2+1} 1^{m_1})} & \text{if } \lambda_3' + 1 \leq k \leq \lambda_2', \\ B_{(3^{m_3-1} 2^{m_2+1} 1^{m_1})} & \text{if } \lambda_2' + 1 \leq k \leq \lambda_1', \\ \begin{array}{l} B_{(3^{m_3-1} 2^{m_2+1} 1^{m_1})} \\ \quad + \frac{1}{2} B_{(3^{m_3-1} 2^{m_2} 1^{m_1+2})} \end{array} & \text{if } \lambda_1' + 1 \leq k \leq N, \end{cases}$$

and Lemma 1 gives

$$(26) \qquad A_{(3^{m_3} 2^{m_2} 1^{m_1})}(k, 2) = \begin{cases} -\frac{1}{2} B_{(3^{m_3-2} 2^{m_2+2} 1^{m_1})} & \text{if } 2 \leq k \leq \lambda_3', \\ 0 & \text{if } \lambda_3' + 1 \leq k \leq \lambda_2', \\ \frac{1}{2} B_{(3^{m_3-1} 2^{m_2} 1^{m_1+1})} & \text{if } \lambda_2' + 1 \leq k \leq \lambda_1', \\ B_{(3^{m_3-1} 2^{m_2} 1^{m_1+1})} & \text{if } \lambda_1' + 1 \leq k \leq N, \end{cases}$$

$$(27) \qquad A_{(3^{m_3} 2^{m_2} 1^{m_1})}(k, 1) = \begin{cases} -B_{(3^{m_3-2} 2^{m_2+1} 1^{m_1+1})} & \text{if } 2 \leq k \leq \lambda_3', \\ -\frac{1}{2} B_{(3^{m_3-1} 2^{m_2-1} 1^{m_1+2})} & \text{if } \lambda_3' + 1 \leq k \leq \lambda_2', \\ 0 & \text{if } \lambda_2' + 1 \leq k \leq \lambda_1', \\ \frac{1}{2} B_{(3^{m_3-1} 2^{m_2} 1^{m_1})} & \text{if } \lambda_1' + 1 \leq k \leq N, \end{cases}$$

and

$$(28) \qquad A_{(3^{m_3} 2^{m_2} 1^{m_1})}(k, 0) = \begin{cases} \begin{array}{l} -B_{(3^{m_3-2} 2^{m_2+1} 1^{m_1})} \\ \quad -\frac{1}{2} B_{(3^{m_3-2} 2^{m_2} 1^{m_1+2})} \end{array} & \text{if } 2 \leq k \leq \lambda_3', \\ -B_{(3^{m_3-1} 2^{m_2-1} 1^{m_1+1})} & \text{if } \lambda_3' + 1 \leq k \leq \lambda_2', \\ -\frac{1}{2} B_{(3^{m_3-1} 2^{m_2} 1^{m_1-1})} & \text{if } \lambda_2' + 1 \leq k \leq \lambda_1', \\ 0 & \text{if } \lambda_1' + 1 \leq k \leq N. \end{cases}$$

Concerning integrals (13) we can prove the following two lemmas.

Lemma 5. *Let λ be any partition with $\ell_\lambda \leq N$. Thus, for $0 \leq c \leq \lambda_1$,*

$$(29) \qquad K_\lambda(c) = K_\lambda(0) - \sum_{i=0}^{c-1} B_{[\lambda+(i-\lambda_1)e_1]}.$$

Proof. The proof of the lemma follows straightforwardly after using the substitution

$$(30) \qquad \frac{y_1^c}{1-y_1} = \frac{1}{1-y_1} - \sum_{i=0}^{c-1} y_1^i$$

in Eq. (13). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 6. *Let λ be any partition with $\ell_\lambda \leq N$. For $0 \leq c \leq \lambda_1$ we have*

$$(31) \qquad K_\lambda(c) = \frac{2\rho}{b-1} \sum_{k=2}^{N} A_\lambda(k,c) + \frac{a-1+c}{b-1} B_{[\lambda+(c-1-\lambda_1)e_1]}.$$

Proof. Since $\Phi(y)$ vanishes at the boundary values $y_1 = 0$ and $y_1 = 1$, we obtain after applying Stokes' theorem,

$$
\begin{aligned}
(32) \qquad 0 &= \int_\Lambda d_1 \left(y_1^c \prod_{j=2}^{\ell} y_j^{\lambda_j} \Phi(y) dy' \right) \\
&= 2\rho \sum_{k=2}^{N} A_\lambda(k,c) + (a-1+c) B_{[\lambda+(c-1-\lambda_1)e_1]} - (b-1) K_\lambda(c),
\end{aligned}
$$

which follows from the fact that

$$(33) \qquad d_1 \Phi(y) = \frac{a}{y_1} - \frac{b-1}{1-y_1} + 2\rho \sum_{k=2}^{N} \frac{1}{y_1 - y_k}.$$

Eq. (31) follows from (33). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The following example essentially reproduces the derivation of the recurrence found in [11] for Selberg integrals involving symmetric monomial polynomials associated to partitions with entries ≤ 2, namely, [11, Lemma 4].

Example 7. *Let $\lambda = (2^{m_2} 1^{m_1})$ and $c = 2$. Lemma 1 gives*

$$(34) \qquad (b-1) K_{(2^{m_2} 1^{m_1})}(2) = 2\rho \sum_{k=2}^{N} A_\lambda(k,2) + (a+1) B_{(2^{m_2-1} 1^{m_1+1})}.$$

By virtue of Example 3 we get

$$(35) \qquad (b-1) K_{(2^{m_2} 1^{m_1})}(2) = (a+1+\rho(2N - m_1 - 2m_2)) B_{(2^{m_2-1} 1^{m_1+1})}.$$

Using Lemma 5 we find

$$
\begin{aligned}
(36) \qquad (b-1) &\left(K_{(2^{m_2} 1^{m_1})}(0) - B_{(2^{m_2-1} 1^{m_1})} - B_{(2^{m_2-1} 1^{m_1+1})} \right) \\
&= (a+1+\rho(2N - 2m_2 - m_1)) B_{(2^{m_2-1} 1^{m_1+1})},
\end{aligned}
$$

which is equivalent to

$$(37) \qquad (b-1)K_{(2^{m_2}1^{m_1})}(0) = (a+b+\rho(2N-m_1-2m_2))B_{(2^{m_2-1}1^{m_1+1})}$$
$$+(b-1)B_{(2^{m_2-1}1^{m_1})},$$

as proved in [11, lemma 2].

In a similar way, for $c = 1$, Lemma 1 and Lemma 5 give

$$(38) \qquad (b-1)\left(K_{(2^{m_2}1^{m_1})}(0) - B_{(2^{m_2-1}1^{m_1})}\right) = 2\rho \sum_{k=2}^{N} A_\lambda(k,1) + aB_{(2^{m_2-1}1^{m_1})},$$

which is equivalent to

$$(39) \qquad (b-1)K_{(2^{m_2}1^{m_1})}(0) = -\rho(m_2-1)B_{(2^{m_2-2}1^{m_1+2})}$$
$$+(a+b-1+\rho(N-m_1-m_2))B_{(2^{m_2-1}1^{m_1})}.$$

After combining (37) and (39) we obtain

$$(40)\ (a+b+\rho(2N-2m_2-m_1))B_{(2^{m_2-1}1^{m_1+1})} = (a+\rho(N-m_1-m_2))B_{(2^{m_2-1}1^{m_1})}$$
$$-\rho(m_2-1)B_{(2^{m_2-2}1^{m_1+2})},$$

as it is proved in [11, Lemma 4, Equation (13)].

3. Recurrence relations for Selberg-type integrals

Formula (40) defines a recursive relation that was used in [11] for computing Selberg-Schur integrals associated to partitions of the form $\lambda = (2^{m_2}1^{m_1})$, $0 \le m_1 + m_2 \le N$. In this section we find a set of recurrence relations satisfied by Selberg integrals involving monomial polynomials generalizing [11, Lemma 4, Equation (13)].

Theorem 8. *Let λ be any partition of length $\ell_\lambda < N$. Then, for any c such that $0 \le c < \lambda_1$ we have*

$$(41) \qquad (b-1)\sum_{i=c}^{\lambda_1-1} B_{[\lambda+(i-\lambda_1)e_1]} + (a-1+\lambda_1)B_{[\lambda-e_1]} - (a-1+c)B_{[\lambda+(c-1-\lambda_1)e_1]}$$

$$= \rho\sum_{k=2}^{N}(-1)^{\delta_{\lambda_k<c}} \sum_{i=1}^{\max\{\lambda_k,c\}-\min\{\lambda_k,c\}} B_{[\lambda+(\max\{\lambda_k,c\}-i-\lambda_1)e_1+(\min\{\lambda_k,c\}+i-1-\lambda_k)e_k]}$$

$$-\rho\sum_{k=2}^{N}\sum_{i=1}^{\lambda_1-\lambda_k} B_{[\lambda-ie_1+(i-1)e_k]},$$

where $\delta_{a<b}$ equals 0 if $a < b$ and it equals 1 otherwise.

Remark 9. *Before proving the Theorem let us emphasize that (41) is actually a well defined recurrence for $c < \lambda_1$ for the dominance ordering on partitions, namely, all partitions appearing in (41) are $\preceq \lambda$.*

Proof. The proof of the theorem follows the same steps as Example 7. Using Lemma 5 for $c = \lambda_1$ we obtain

$$(42) \qquad (b-1)\left(K_\lambda(0) - \sum_{i=0}^{\lambda_1-1} B_{[\lambda+(i-\lambda_1)e_1]}\right) = 2\rho\sum_{k=2}^{N} A_\lambda(k) + (a-1+\lambda_1)B_{[\lambda-e_1]},$$

which is equivalent to

$$(43) \qquad (b-1)K_\lambda(0) = (b-1)\sum_{i=0}^{\lambda_1-1} B_{[\lambda+(i-\lambda_1)e_1]} + (a-1+\lambda_1)B_{[\lambda-e_1]}$$

$$+\rho\sum_{k=2}^{N}\sum_{i=1}^{\lambda_1-\lambda_k} B_{[\lambda-ie_1+(i-1)e_k]}.$$

On the other hand, Lemma 5 for an arbitrary $0 \le c < \lambda_1$ gives

$$(44) \quad (b-1)\left(K_\lambda(0) - \sum_{i=0}^{c-1} B_{[\lambda+(i-\lambda_1)e_1]}\right) = 2\rho\sum_{k=2}^{N} A_\lambda(k,c) + (a-1+c)B_{[\lambda+(c-1-\lambda_1)e_1]},$$

and by Lemma 1 it follows that

$$(45) \qquad \sum_{k=2}^{N} A_\lambda(k,c) = -\frac{1}{2}\sum_{k=2,c\le\lambda_k}^{N}\sum_{i=0}^{\lambda_k-1-c} B_{[\lambda+(c+i-\lambda_1)e_1-(1+i)e_k]}$$

$$+\frac{1}{2}\sum_{k=2,c>\lambda_k}^{N}\sum_{i=0}^{c-\lambda_k-1} B_{[\lambda+(c-1-i-\lambda_1)e_1+ie_k]},$$

so that

$$(46) \qquad (b-1)K_\lambda(0) = (b-1)\sum_{i=0}^{c-1} B_{[\lambda+(i-\lambda_1)e_1]} + (a-1+c)B_{[\lambda+(c-1-\lambda_1)e_1]}$$

$$-\rho\sum_{k=2,c\le\lambda_k}^{N}\sum_{i=0}^{\lambda_k-1-c} B_{[\lambda+(c+i-\lambda_1)e_1-(1+i)e_k]} + \rho\sum_{k=2,c>\lambda_k}^{N}\sum_{i=0}^{c-\lambda_k-1} B_{[\lambda+(c-1-i-\lambda_1)e_1+ie_k]}.$$

After combining Equations (43) and (46) we get the desired result.   □

Corollary 10. *Let λ be a partition of length $\ell_\lambda < N$ with $\lambda_1 > \lambda_k$ for $k = 2, \ldots, N$. Then,*

$$(47) \quad (a+b+\lambda_1-2)B_{[\lambda-e_1]} - (a+\lambda_1-2)B_{[\lambda-2e_1]} = -\rho\sum_{k=2}^{N}\sum_{i=1}^{\lambda_1-\lambda_k} B_{[\lambda-ie_1+(i-1)e_k]}$$

$$+\rho\sum_{k=2}^{N}\sum_{i=1}^{\lambda_1-\lambda_k-1} B_{[\lambda-(i+1)e_1+(i-1)e_k]}.$$

Proof. The proof of the corollary follows straightforwardly after replacing $c = \lambda_1 - 1$ in Eq. (41).   □

In the next section we give some examples showing the usefulness of these results.

4. Applications

Let us recall that any Schur polynomial can be expressed in term of monomial symmetric polynomials as

$$(48) \qquad s_\lambda(y) = \sum_{\mu \preceq \lambda} K_{\lambda\mu} m_\mu(y),$$

where $K_{\lambda\mu}$ is the Kostka number associated with λ and μ and $m_\mu(y)$ is the monomial symmetric polynomial indexed by μ.

From the symmetry of the Selberg-Schur kernel under the permutation of any pair of variables it follows that

$$(49) \qquad J_\lambda = \sum_{\mu \preceq \lambda} m_\mu(1^N) K_{\lambda\mu} B_\mu,$$

where we have exploit that $m_\mu(y)$ is a symmetric polynomial, thus proving that Theorem 8 provides us with a well-defined algorithm for computing Selberg-Schur integral as it was announced in the introduction.

Let us illustrate this fact with some interesting examples.

4.1. **Partitions of the form** (31^n)**:** For a given positive integer n let us consider the partition (31^n). The partitions μ satisfying that $\mu \preceq (31^n)$ are: (1^{n+3}), (21^{n+1}) and (31^n). Recall from [1] that

$$(50) \qquad B_{(1^n)} = J_0 \frac{[a + (N-1)\rho]_{(1^n)}^{(\rho)}}{[a + b + 2(N-1)\rho]_{(1^n)}^{(\rho)}},$$

and from [11] that

$$(51) \qquad \begin{aligned} B_{(2^n 1^m)} = J_0 \frac{[a + (N-1)\rho]_{(2^n 1^m)}^{(\rho)}}{[a + b + 2(N-1)\rho]_{(2^n 1^m)}^{(\rho)}} \, \frac{[a + b + (N-2)\rho]_{(1^n)}^{(\rho)}}{[a + b + (2N - m - n - 2)\rho]_{(1^n)}^{\rho}} \\ \times {}_3F_2\left[\begin{matrix} -n, -N + m + n, \alpha + \beta + \gamma + 2N - n - 1 \\ \alpha + \beta + N - n - 1, \alpha + \gamma + N - n \end{matrix} \right], \end{aligned}$$

for any positive integer m, where, as before, $\alpha = a/\rho$, $\beta = b/\rho$, $\gamma = 1/\rho$ and the hypergeometric series ${}_3F_2$ is evaluated at 1.

Noticing that

$$(52) \qquad K_{(31^n),(1^{n+3})} = \frac{1}{2} \frac{(n+2)!}{n!},$$

$$(53) \qquad K_{(31^n),(21^{n+1})} = n + 1,$$

it follows from Eq. (49) that in order to find an expression for $J_{(31^n)}$ we only need to find an explicit formula for $B_{(31^n)}$.

Applying Eq. (41) for $\lambda = (41^n)$ and $c = 2$ we obtain

$$(54) \qquad \begin{aligned} \rho \big[2(N - n - 1)\left(B_{(1^{n+1})} - B_{(31^n)} - B_{(21^{n+1})}\right) - n\left(2B_{(31^n)} + B_{(2^2 1^{n-1})}\right) \\ + nB_{(1^{n+1})} \big] = (b-1)B_{(21^n)} + (b-1)B_{(31^n)} + (a+3)B_{(31^n)} - (a+1)B_{(1^{n+1})}, \end{aligned}$$

namely,

$$(55) \qquad B_{(31^n)} = [a + b + 2 + 2\rho(N-1)]^{-1} \times \big[(a + 1 + \rho(2N - n - 2))B_{(1^{n+1})}$$
$$- (b-1)B_{(21^n)} - 2\rho(N - n - 1)B_{(21^{n+1})} - \rho n B_{(2^2 1^{n-1})} \big].$$

Substitution of (50) and (51) into this last expression will eventually lead us to the desired formula for $B_{(31^n)}$.

4.2. **Partitions of the form** (32^m)**:** Let m be, again, a non-negative integer number and let us consider the case of the partition (32^m). Despite of the lack of explicit expressions for the corresponding Kostka numbers, it is straightforward to find a recurrence for $B_{(32^m)}$.

In fact, applying Theorem 8 for $\lambda = (42^m)$ with $c = 3$ we have that

$$(56) \qquad B_{(32^m)} = [a + b + 2 + 2\rho(N-1)]^{-1} \big[(a + 2 + \rho(2N - m - 2))B_{(2^{m+1})}$$
$$+ \rho(N - m - 1)B_{(2^m 1^2)} - 2\rho(N - m - 1)B_{(2^{m+1}1)} \big],$$

from where a formula for $B_{(32^m)}$ can be read.

4.3. **Partitions of the form** $(32^m 1^n)$**:** Finally let us discuss the case of partitions of the form $(32^m 1^n)$ with $m, n > 0$. Applying again Theorem 8 but now for $\lambda = (42^m 1^n)$ and $c = 3$ we get:

$$(57) \qquad B_{(32^m 1^n)} = [a + b + 2 + 2\rho(N-1)]^{-1} \big[(a + 2 + \rho(2N - m - n - 2))B_{(2^{m+1}1^n)}$$
$$+ \rho(N - m - n - 1)B_{(2^m 1^{n+2})} - n\rho B_{(2^{m+2}1^{n-1})} - 2\rho(N - m - n - 1)B_{(2^{m+1}1^{n+1})} \big].$$

As before, using (50) and (51) an explicit formula for $B_{(32^m 1^n)}$ can be derived.

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